HAZARD RATE PROPERTIES FOR THE WARM STANDBY SYSTEM

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THESIS

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by

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Hazard Rate Properties
for the
Warm Standby System

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ABSTRACT

Properties of the hazard rate function are examined for a warm standby system composed of one main component with an exponentially distributed life length and one standby component with a two-stage exponentially distributed life length. Emphasis is placed on comparing the warm standby hazard rate function with the hazard rate functions for the parallel and cold standby systems.

Graphical analysis suggests certain hazard rate properties, some of which are proved while others are conjectures. Properties with possible applications examined include initial and terminal values, monotonicity, and choice of component for the active role. Areas requiring further investigation are noted.

TABLE OF CONTENTS

LIST OF DRAWINGS

I. INTRODUCTION

Consider a system composed of two independent components each of which performs the same task. In order for the system to complete its assigned mission either one or both components must function throughout the mission. Each component is assumed to have a random life length. A practical example might be a shipboard steam generator and an emergency diesel generator.

The two-component system may operate in one of three modes:

1. Parallel
2. Cold standby
3. Warm standby

In the parallel mode both components function at full capacity simultaneously. In the cold standby mode component two does not begin to function until component one fails. The warm standby mode is the intermediate case where component two operates at a fraction of its capacity until unit one fails at which time it becomes fully operational.

The hazard rate function is an informative and intuitively appealing characterization of life distributions. Properties of the cold standby hazard rate function were studied by Barlow, Marshall, and Proschan [Ref. 1] when the components have increasing failure rate distributions. The hazard rate function for the parallel system has been studied by Esary and Proschan [Ref. 4] and by Bryson

[Ref. 3] when the components have exponential distributions, and by Birnbaum, Esary, and Marshall [Ref. 2] when the components have increasing failure rate average distributions.

The concern in this thesis is to examine the properties of the hazard rate functions of the three system modes. Component one is assumed to have an exponential life distribution. Component two is assumed to have a two-stage exponential life distribution where the first stage failure rate is a fraction of the second stage failure rate.

II. DERIVING THE HAZARD RATE

A. THE SURVIVAL FUNCTION

The survival function is defined as the complement of the usual distribution function. Let $T \geq 0$ represent the random time to failure of the system. The survival function is then $\bar{F}(t) = P(T > t)$, $t \geq 0$.

For the systems considered herein, let T_i represent the time to failure of component i, for i = 1, 2. Assume T_1 has an exponential distribution with failure rate λ_1. Assume T_2 has the failure rate $\theta\lambda_2$ for all $t \leq T_1$ and the failure rate λ_2 for all $t > T_1$ where $0 \leq \theta \leq 1$. The situation is illustrated in Figure 1.

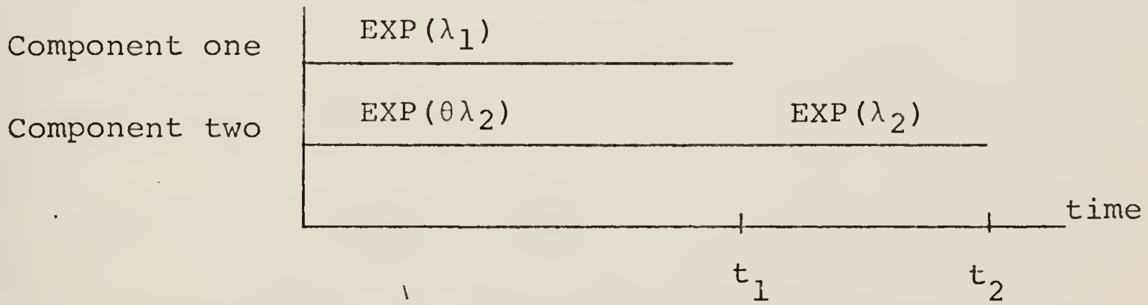

FIGURE 1. LIFE DISTRIBUTION OF T

For the system to survive a mission of duration t, either component one must survive to time t, or fail at some time $s \leq t$ while component two survives to time s and then on to time t. Then,

$$\bar{F}(t) = e^{-\lambda_1 t} + \int_0^t e^{-\theta\lambda_2 s}\, e^{-\lambda_2 (t-s)}\, \lambda_1 e^{-\lambda_1 s}\, ds$$

$$= e^{-\lambda_1 t} + e^{-\lambda_2 t} \int_0^t \lambda_1 e^{-(\lambda_1 - (1-\theta)\lambda_2)s}\, ds, \quad t \geq 0.$$

The derivation proceeds in two cases:

Case I. If $\lambda_1 \neq (1-\theta)\lambda_2$, then

$$\bar{F}(t) = \frac{(\lambda_1-(1-\theta)\lambda_2)e^{-\lambda_1 t}+\lambda_1 e^{-\lambda_2 t}-\lambda_1 e^{-(\lambda_1+\theta\lambda_2)t}}{\lambda_1-(1-\theta)\lambda_2}, \quad t \geq 0.$$

Case II. If $\lambda_1 = (1-\theta)\lambda_2$, then

$$\bar{F}(t) = e^{-\lambda_1 t} + \lambda_1 t e^{-\lambda_2 t}, \quad t \geq 0.$$

This survival function is applicable for all three modes of the system. That is, for $\theta = 0$, the cold standby survival function is Case I ($\lambda_1 \neq \lambda_2$).

$$\bar{F}(t) = \frac{\lambda_1 e^{-\lambda_2 t} - \lambda_2 e^{-\lambda_1 t}}{\lambda_1 - \lambda_2}, \quad t \geq 0,$$

Case II ($\lambda_1 = \lambda_2 = \lambda$).

$$\bar{F}(t) = e^{-\lambda t} (1+\lambda t), \quad t \geq 0.$$

For $\theta =1$, the parallel mode survival function is

$$\bar{F}(t) = e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)t}, \quad t \geq 0.$$

B. THE HAZARD RATE

The hazard rate function, $r(t)$, is defined as the negative derivative of the survival function divided by the survival function. That is, $r(t) = f(t)/\bar{F}(t)$, where $f(t) = -d\bar{F}(t)/dt$ is the density function. For all $0 \leq \theta \leq 1$ the hazard rate is

Case I.

$$f(t) = \frac{\lambda_1(\lambda_1-(1-\theta)\lambda_2)e^{-\lambda_1 t}+\lambda_1\lambda_2 e^{-\lambda_2 t}-\lambda_1(\lambda_1+\theta\lambda_2)e^{-(\lambda_1+\theta\lambda_2)t}}{\lambda_1-(1-\theta)\lambda_2}, \quad t\geq 0,$$

$$r(t) = \frac{\lambda_1(\lambda_1-(1-\theta)\lambda_2)e^{-\lambda_1 t}+\lambda_1\lambda_2 e^{-\lambda_2 t}-\lambda_1(\lambda_1+\theta\lambda_2)e^{-(\lambda_1+\theta\lambda_2)t}}{(\lambda_1-(1-\theta)\lambda_2)e^{-\lambda_1 t}+\lambda_1 e^{-\lambda_2 t}-\lambda_1 e^{-(\lambda_1+\theta\lambda_2)t}}, \quad t\geq 0.$$

Case II.

$$f(t) = \lambda_1 e^{-\lambda_1 t}+\lambda_1\lambda_2 te^{-\lambda_2 t}-\lambda_1 e^{-\lambda_2 t}, \quad t\geq 0,$$

$$r(t) = \frac{\lambda_1 e^{-\lambda_1 t}+\lambda_1\lambda_2 te^{-\lambda_2 t}-\lambda_1 e^{-\lambda_2 t}}{e^{-\lambda_1 t}+\lambda_1 te^{-\lambda_2 t}}, \quad t\geq 0.$$

For the boundary cases, the hazard rate function for the cold standby system ($\theta=0$) is

Case I ($\lambda_1 \neq \lambda_2$).

$$r(t) = \frac{\lambda_1\lambda_2\ (e^{-\lambda_2 t} - e^{-\lambda_1 t})}{\lambda_1 e^{-\lambda_2 t} - \lambda_2 e^{-\lambda_1 t}}, \quad t\geq 0,$$

Case II ($\lambda_1 = \lambda_2 = \lambda$).

$$r(t) = \frac{\lambda t}{1 + \lambda t}, \quad t\geq 0.$$

For the parallel system ($\theta = 1$) the hazard rate function is

$$r(t) = \frac{\lambda_1 e^{-\lambda_1 t}+\lambda_2 e^{-\lambda_2 t}-(\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)t}}{e^{-\lambda_1 t}+e^{-\lambda_2 t}-e^{-(\lambda_1+\lambda_2)t}}, \quad t\geq 0.$$

III. GRAPHS OF THE HAZARD RATE

Normalization of the failure rates, λ_1 and λ_2, facilitates graphing of the hazard rate function. Let $\alpha_1 = \lambda_1/(\lambda_1+\lambda_2)$ and $\alpha_2 = \lambda_2/(\lambda_1+\lambda_2)$. Then $\alpha_1 + \alpha_2 = 1$. Also, let $\tau = t(\lambda_1+\lambda_2)$. Then

Case I $\alpha_1 \neq (1 - \theta)\alpha_2$.

$$r(\tau) = \frac{\alpha_1(\alpha_1-(1-\theta)\alpha_2)e^{-\alpha_1\tau}+\alpha_1\alpha_2 e^{-\alpha_2\tau}-\alpha_1(\alpha_1+\theta\alpha_2)e^{-(\alpha_1+\theta\alpha_2)\tau}}{(\alpha_1-(1-\theta)\alpha_2)e^{-\alpha_1\tau}+\alpha_1 e^{-\alpha_2\tau}-\alpha_1 e^{-(\alpha_1+\theta\alpha_2)\tau}}, \quad \tau\geq 0,$$

Case II $\alpha_1 = (1 - \theta)\alpha_2$.

$$r(\tau) = \frac{\alpha_1 e^{-\alpha_1\tau}+\alpha_1\alpha_2\tau e^{-\alpha_2\tau}-\alpha_1 e^{-\alpha_2\tau}}{e^{-\alpha_1\tau}+\alpha_1\tau e^{-\alpha_2\tau}}, \quad \tau\geq 0.$$

The normalization described above is equivalent to the change of scale $S = (\lambda_1 + \lambda_2)T$, for which

$$r_S(\tau) = \frac{1}{\lambda_1 + \lambda_2} \, r_T\left(\frac{\tau}{\lambda_1 + \lambda_2}\right), \quad \tau\geq 0.$$

Figures 2.1 through 2.8 are presented to illustrate certain properties and conjectures in the next section. The hazard rate and time have been normalized in accordance with the procedure described above.

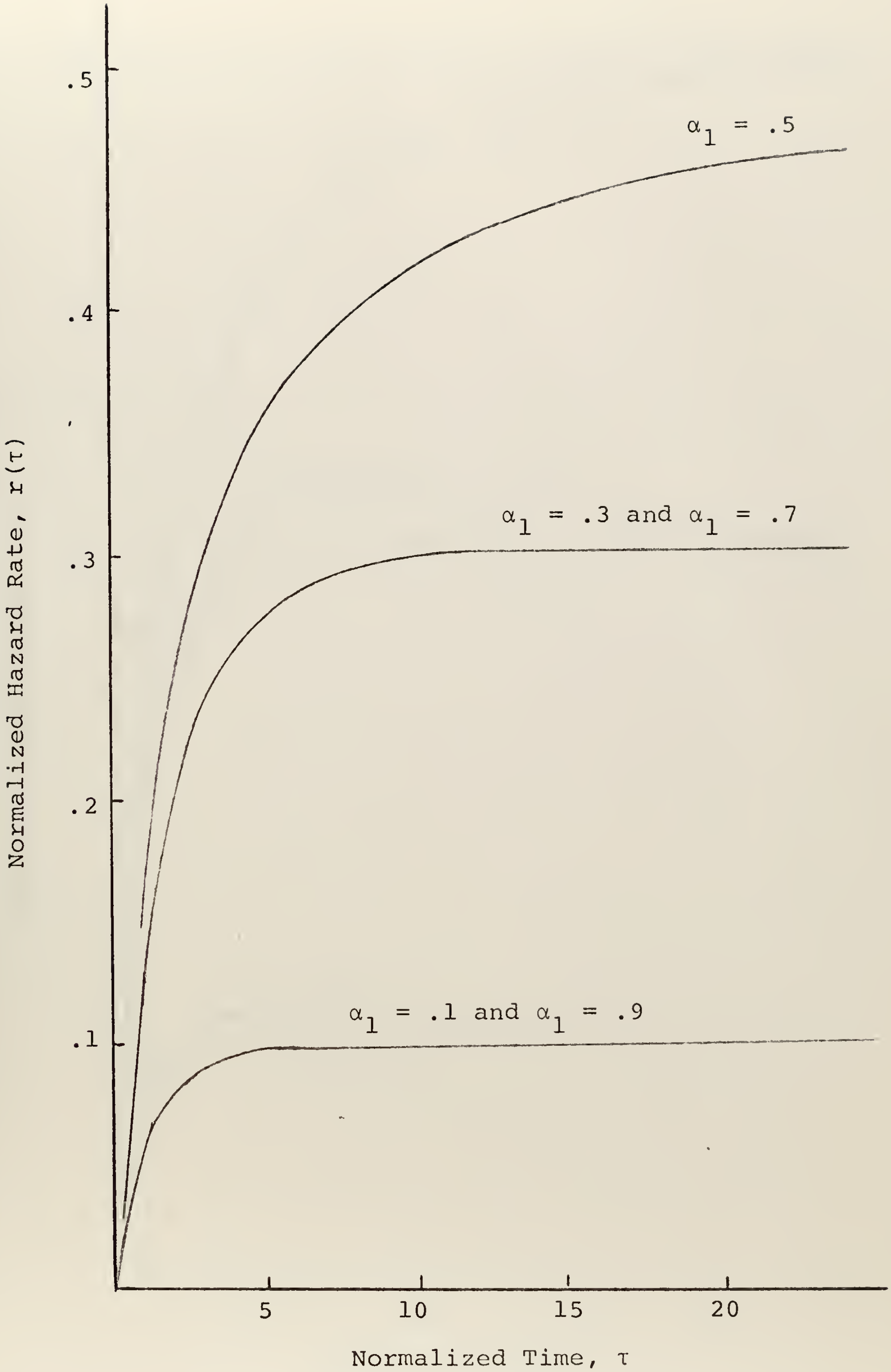

FIGURE 2.1. COLD STANDBY, $\theta = 0$

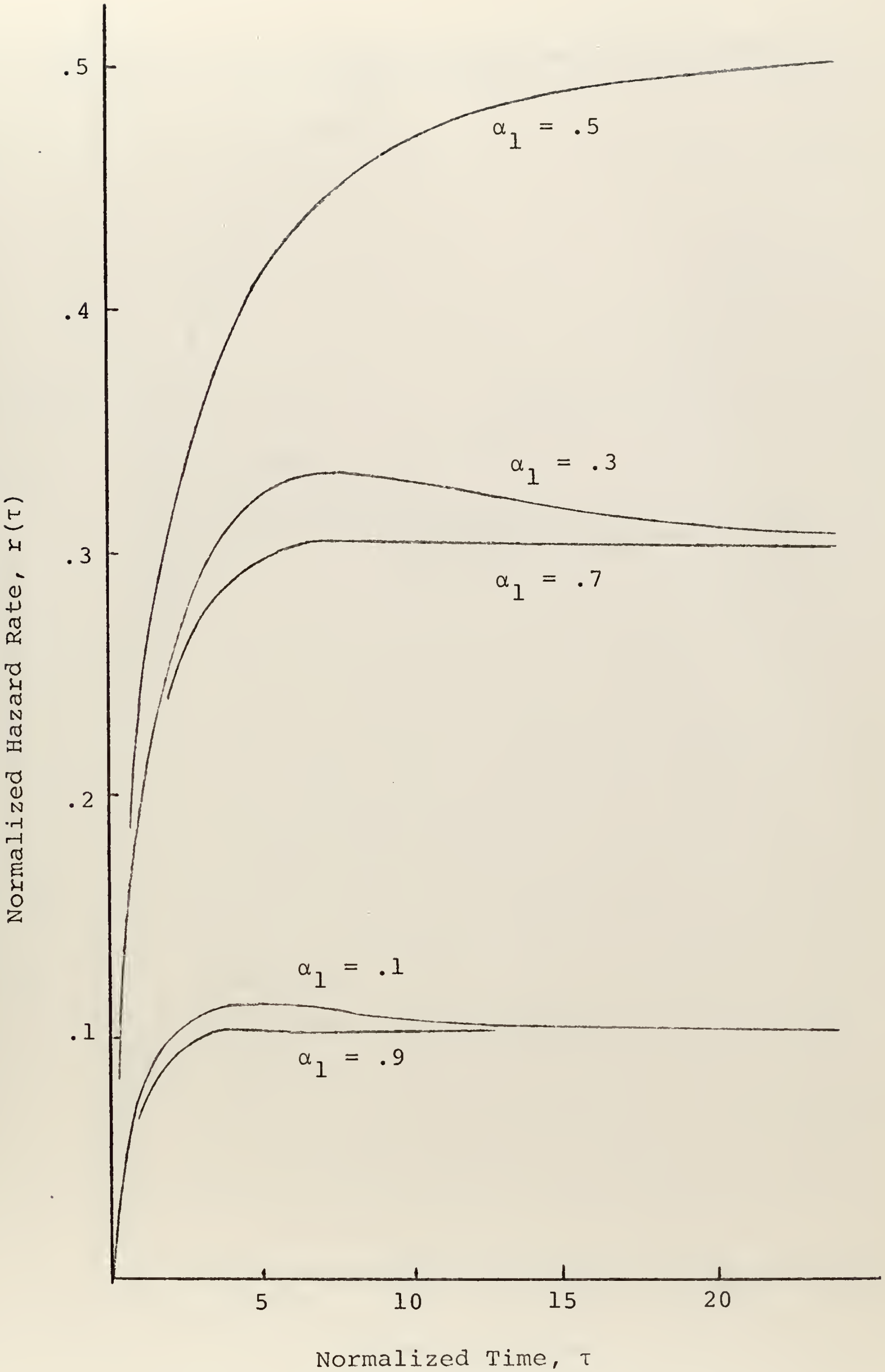

FIGURE 2.2. WARM STANDBY, θ = .25

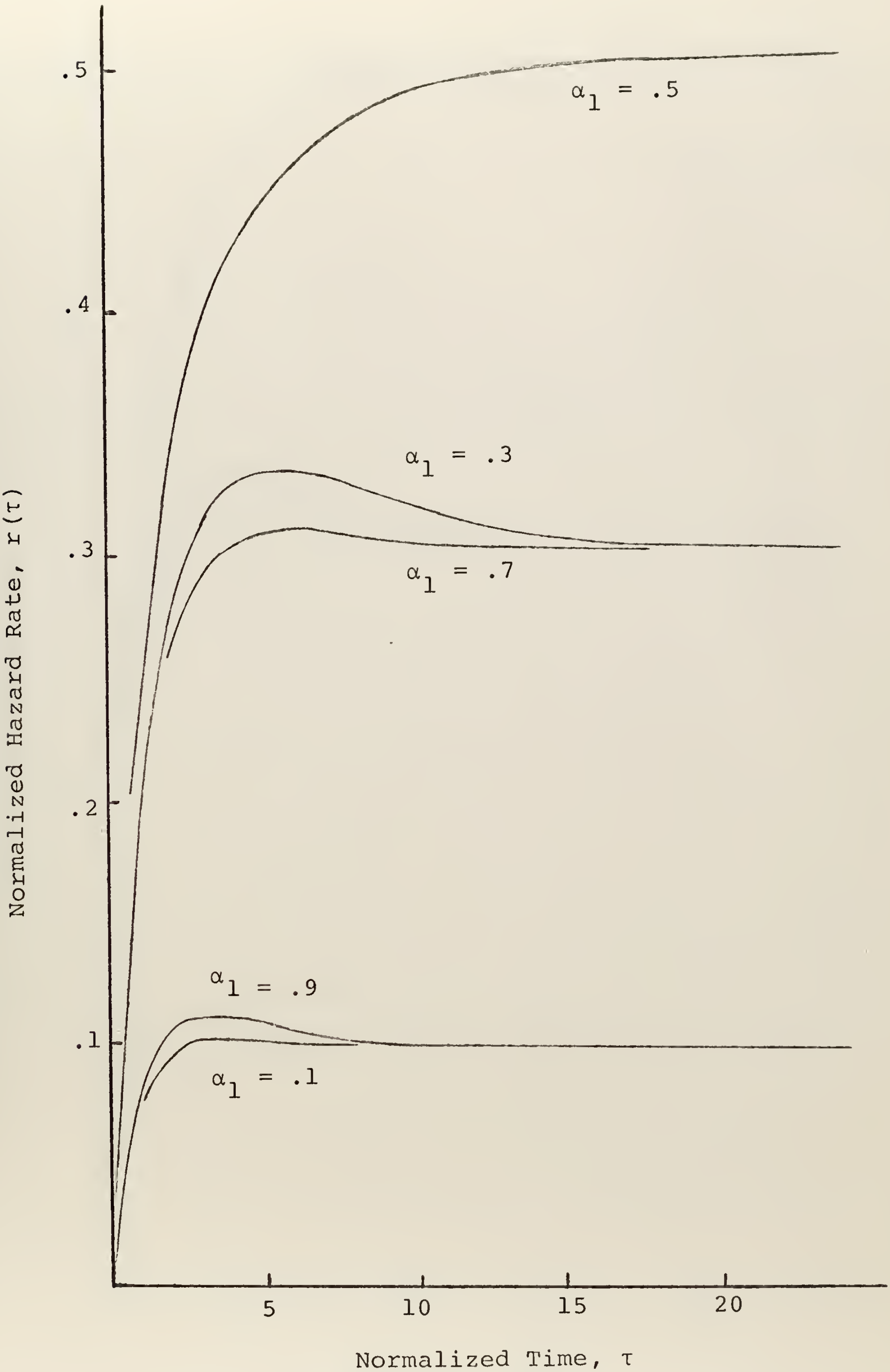

FIGURE 2.3. WARM STANDBY, θ = .5

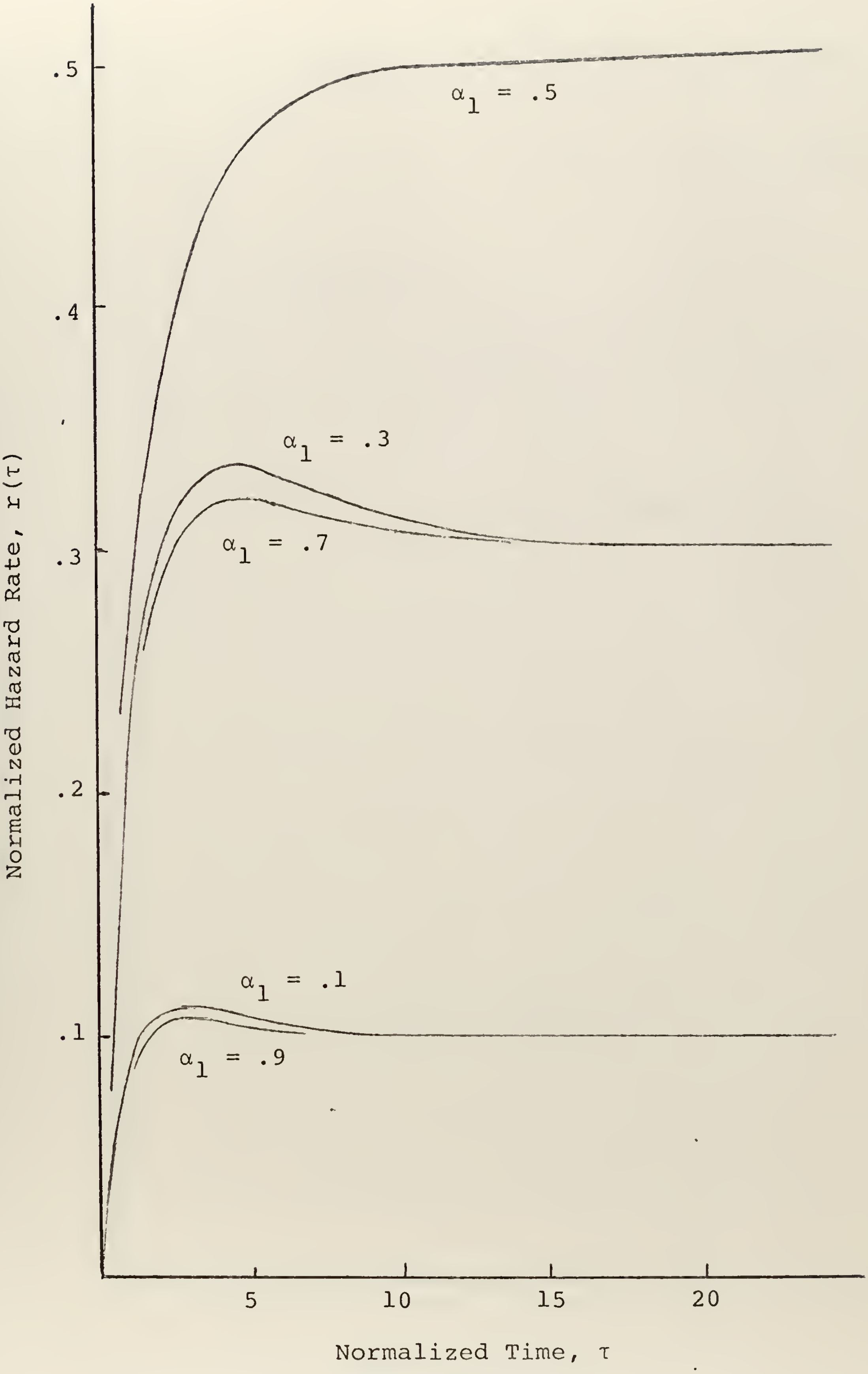

FIGURE 2.4. WARM STANDBY, θ = .75

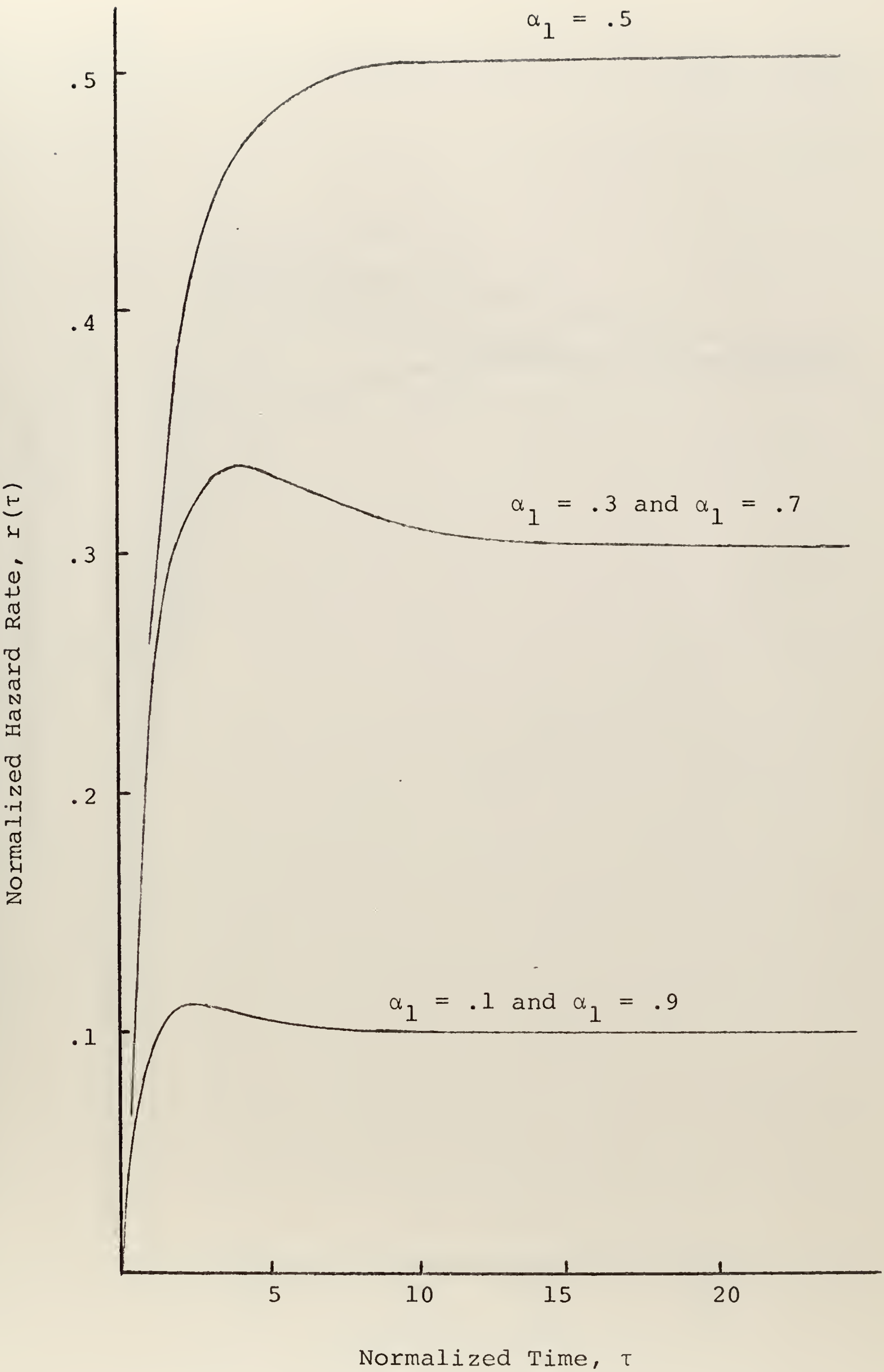

FIGURE 2.5. PARALLEL, $\theta = 1$

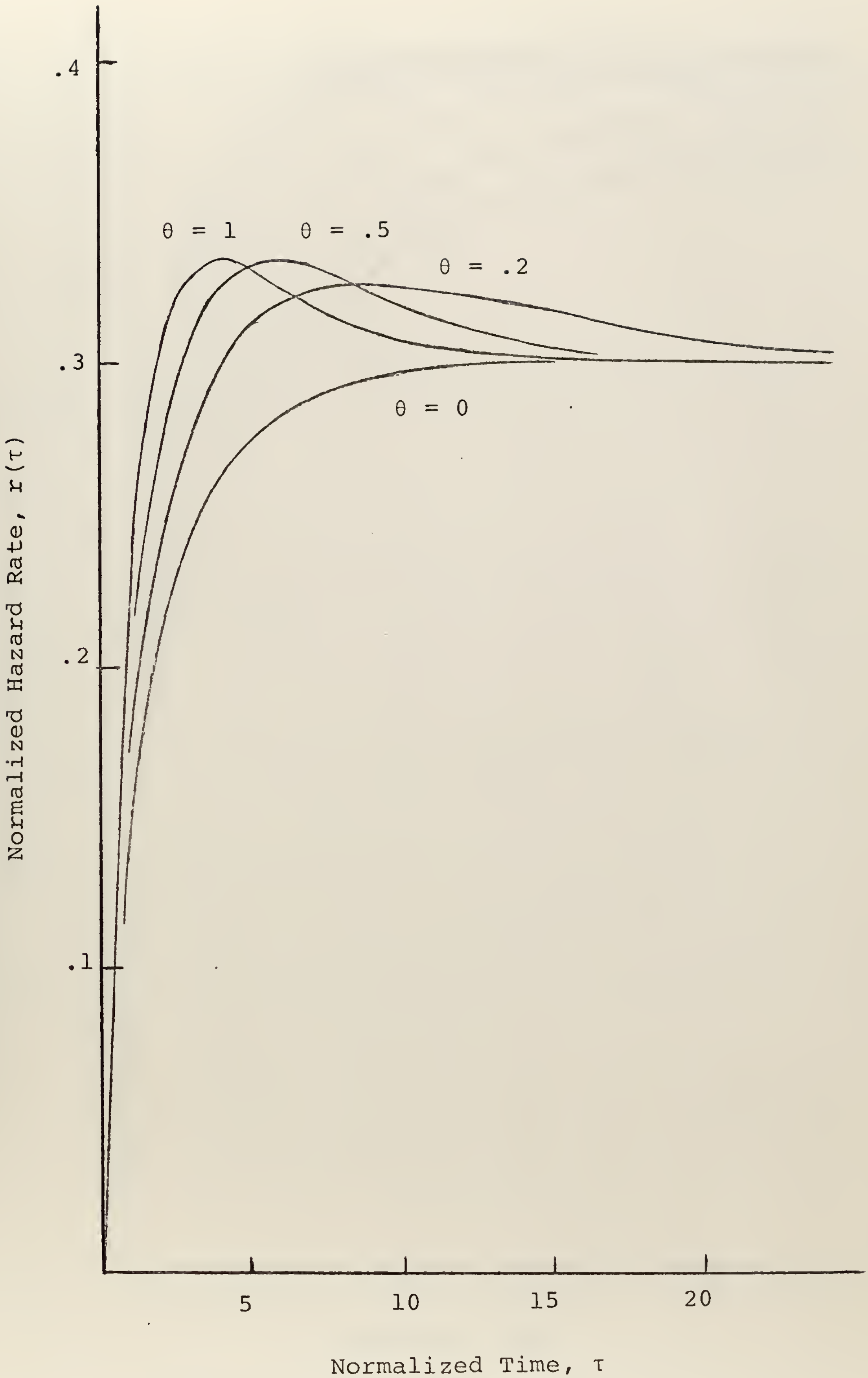

FIGURE 2.6. α_1 = .3, VARYING θ

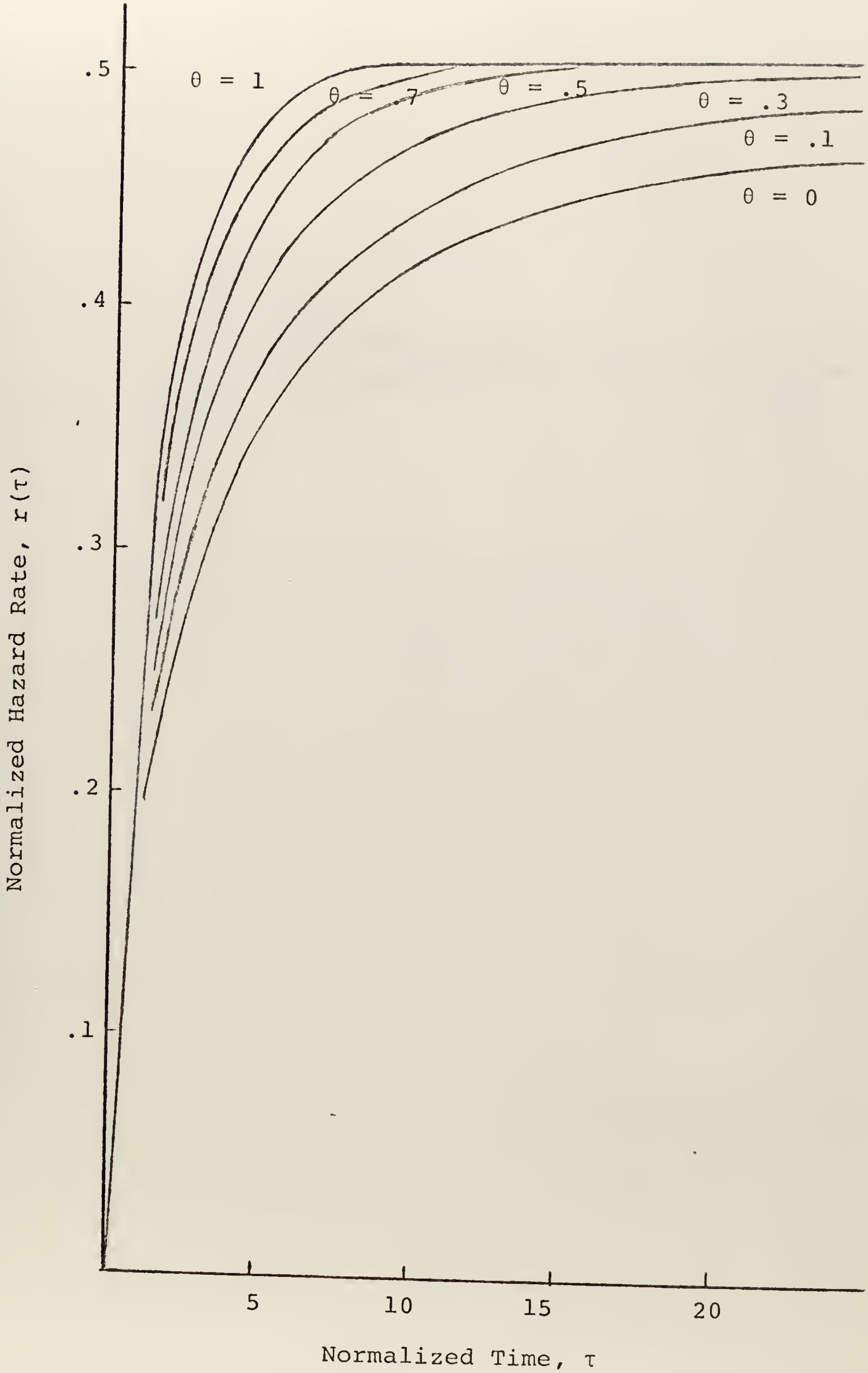

FIGURE 2.7. $\alpha_1 = .5$, VARYING θ

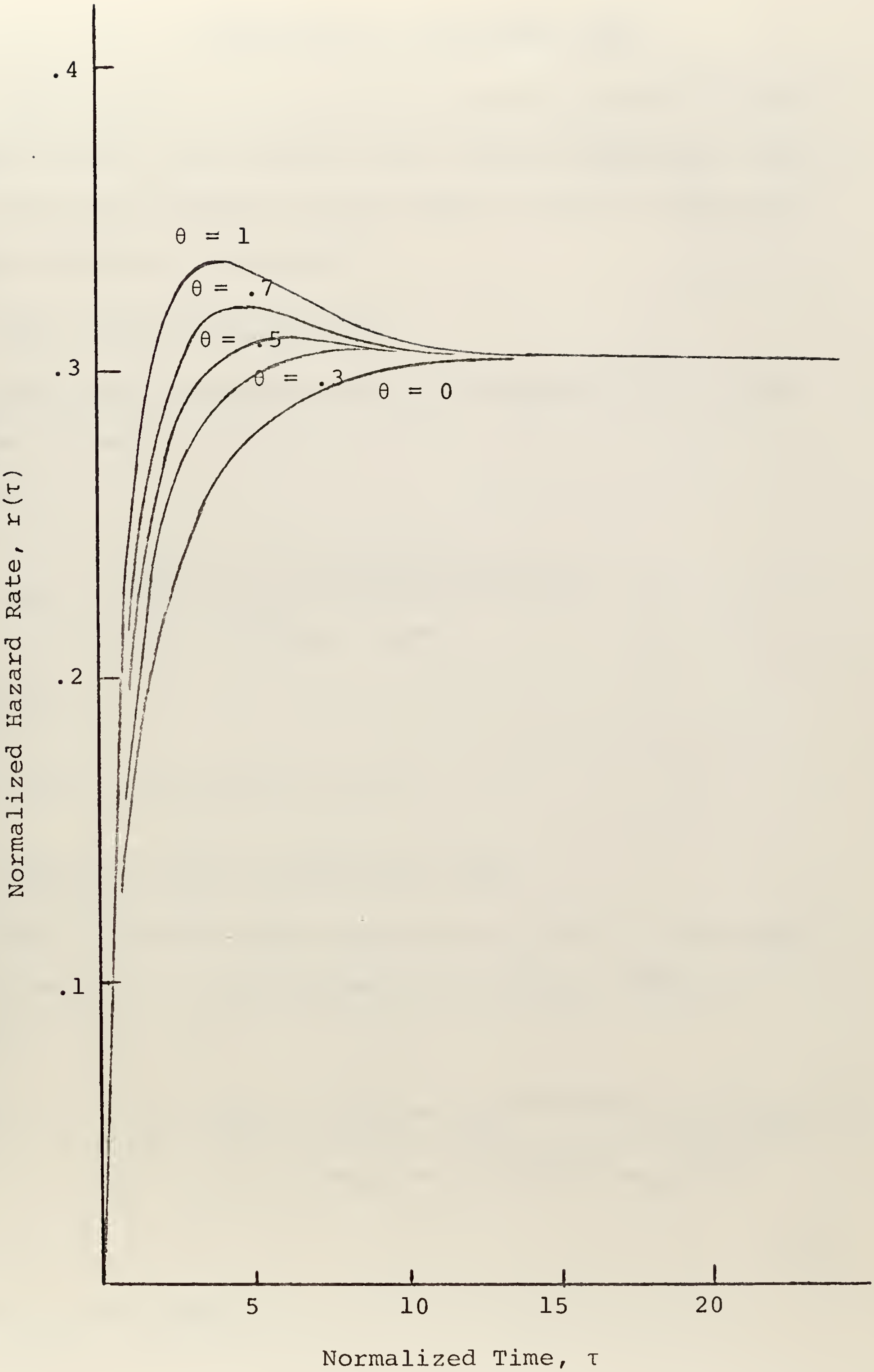

FIGURE 2.8. $\alpha_1 = .7$, VARYING θ

IV. PROPERTIES OF THE HAZARD RATE

The figures of the previous section suggest that the system hazard rate function has certain properties, some of which are subject to easy proof and some of which currently stand as conjectures.

A. THE INITIAL HAZARD RATE

The hazard rate at time $\tau = 0$ is zero for all three modes. That is,

Case I.

$$r(0) = \frac{\alpha_1(\alpha_a-(1-\theta)\alpha_2)+\alpha_1\alpha_2-\alpha_1(\alpha_1+\theta\alpha_2)}{(\alpha_1-(1-\theta)\alpha_2) + \alpha_1-\alpha_1} = 0,$$

Case II.

$$r(0) = \alpha_1+\alpha_1\alpha_2(0)-\alpha_1 = 0.$$

B. TERMINAL VALUE OF THE HAZARD RATE

For all three modes the terminal value of the hazard rate function is the minimum of α_1 and α_2. That is,

Case I. For $\alpha_1<\alpha_2$,

$$\lim_{\tau \to \infty} r(\tau) = \lim_{\tau\to\infty} \frac{\alpha_1(\alpha_1-(1-\theta)\alpha_2)+\alpha_1\alpha_2 e^{-(\alpha_2-\alpha_1)\tau}-\alpha_1(\alpha_1+\theta\alpha_2)e^{-\theta\alpha_2\tau}}{(\alpha_1-(1-\theta)\alpha_2)+\alpha_1 e^{-(\alpha_2-\alpha_1)\tau}-\alpha_1 e^{-\theta\alpha_2\tau}}$$

$$= \alpha_1,$$

Case I. For $\alpha_2<\alpha_1$,

$$\lim_{\tau \to \infty} r(\tau) =$$

$$\lim_{\tau\to\infty} \frac{\alpha_1(\alpha_1-(1-\theta)\alpha_2)e^{-(\alpha_1-\alpha_2)\tau}+\alpha_1\alpha_2-\alpha_1(\alpha_1+\theta\alpha_2)e^{-(\alpha_1-\alpha_2(1-\theta))\tau}}{(\alpha_1-(1-\theta)\alpha_2)e^{-(\alpha_1-\alpha_2)\tau}+\alpha_1-\alpha_1 e^{-(\alpha_1-\alpha_2(1-\theta))\tau}}$$

$$= \alpha_2,$$

Case II. For all $0\leq\theta\leq 1$, $\alpha_1\leq\alpha_2$,

$$\lim_{\tau \to \infty} r(\tau) = \lim_{\tau\to\infty} \frac{\alpha_1+\alpha_1\alpha_2\tau e^{-(\alpha_2-\alpha_1)\tau}-\alpha_1 e^{-(\alpha_2-\alpha_1)\tau}}{1+\alpha_1\tau e^{-(\alpha_2-\alpha_1)\tau}} = \alpha_1.$$

C. MONOTONICITY OF THE HAZARD RATE

For any value of α_1 and α_2 the hazard rate function for the cold standby system ($\theta = 0$) is monotonically increasing. This is easily shown, and is a special case of a theorem proved in [Ref. 1]. If $\alpha_1 = \alpha_2$, the hazard rate of the parallel system ($\theta = 1$) is monotonically increasing. This is also easily shown, and is a special case of a result given in [Ref. 4].

If $\alpha_1 = \alpha_2 = \alpha$, the hazard rate for the warm standby system is also monotonically increasing, since then $\alpha_1 \neq (1 - \theta)\alpha_2$, that is $\theta \neq 1$, and

$$r(\tau) = \frac{(1+\theta)\alpha\ (1 - e^{-\theta\alpha\tau})}{1 + \theta - e^{-\theta\alpha\tau}}, \quad \tau\geq 0,$$

from which

$$\frac{dr(\tau)}{d\tau} = \frac{(1+\theta)\ \theta^2\alpha^2\ e^{-\theta\alpha\tau}}{(1 + \theta - e^{-\theta\alpha\tau})^2}, \quad \geq 0, \text{ for all } \tau\geq 0.$$

This property is illustrated in Figure 2.7.

D. VARIATION OF THE HAZARD RATE WITH θ

Consider two values of θ, $\theta_1 > \theta_2$. One might expect that for fixed α_1 and α_2

$$r_{\theta_1}(\tau) \geq r_{\theta_2}(\tau) \text{ for all } \tau \geq 0.$$

Figure 2.7 and 2.8 where $\alpha_1 = .5$ and $\alpha_1 = .7$ respectively suggest that this is true. However, for $\alpha_1 = .3$ (Fig. 2.6)

$$r_{\theta_1}(\tau) \not\geq r_{\theta_2}(\tau) \text{ for all } \tau \geq 0.$$

However, it appears that for any fixed α_1 and α_2 and any $\theta > 0$

$$r_{\theta}(\tau) \geq r_0(\tau) \text{ for all } \tau \geq 0.$$

This conjecture is suggested by Figs. 2.6, 2.7, and 2.8. Moreover, if $\alpha_1 = \alpha_2 = \alpha$ and $\theta_1 > \theta_2$, then

$$r_{\theta_1}(\tau) \geq r_{\theta_2}(\tau) \text{ for all } \tau \geq 0,$$

since then

$$\frac{\partial r(\tau)}{\partial \theta} = \frac{\alpha e^{-\theta\alpha\tau}\{(1+\theta)\theta\alpha\tau - (1-e^{-\theta\alpha\tau})\}}{\{1 + \theta - e^{-\theta\alpha\tau}\}^2}, \quad \tau \geq 0,$$

and $1 - e^{-\theta\alpha\tau} \leq \theta\alpha\tau \leq (1+\theta)\theta\alpha\tau$ so that $\frac{\partial r(\tau)}{\partial \theta} \geq 0$, for all $\tau \geq 0$.

E. INTERCHANGEABLE COMPONENTS

Consider two components with failure rates α_A and α_B.

$$r(\tau)\alpha_1 = \alpha_A,\ \alpha_2 = \alpha_B = r(\tau)\ \alpha_1 = \alpha_B,\ \alpha_2 = \alpha_A$$

if $\theta = 0$ or $\theta = 1$. For the cold standby and parallel systems interchanging the two components does not change the hazard rate function (Figs. 2.1 and 2.5).

If $\alpha_A \geq \alpha_B$ for any $0 \leq \theta \leq 1$, it appears that

$$r(\tau)\alpha_1 = \alpha_A,\ \alpha_2 = \alpha_B \leq r(\tau)\ \alpha_1 = \alpha_B,\ \alpha_2 = \alpha_A \text{ for all } \tau \geq 0.$$

For the warm standby mode, letting component one have the maximum failure rate of the two components seems to yield a lower system hazard rate for all τ than letting component two have the maximum failure rate. This conjecture is suggested by Figs. 2.2, 2.3, and 2.4.

F. SUMMARY

The cold standby and parallel hazard rate functions have been studied previously and many of their properties have been proven and are generally known.

Because of the more complex nature of the warm standby hazard rate function some of the properties presented herein are conjectures based on graphical analysis. Further investigation of these conjectures is needed. In addition, determining if the warm standby hazard rate function has an increasing hazard rate average for all values of θ, α_1, and α_2 would be particularly interesting. It is known that this is the case for the parallel system [Ref. 2].

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